

# A New Window Onto Quantum Chaos

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In this article the statistical properties of symmetrical random matrices whose elements are drawn from a  $q$ -parameterized non-extensive statistics power-law distribution are investigated. In the limit as  $q \rightarrow 1$  the well known Gaussian orthogonal ensemble (GOE) results are recovered. The relevant level spacing distribution is derived and one obtains a suitably generalized non-extensive Wigner distribution which depends on the value of the tunable non-extensivity parameter  $q$ . This non-extensive Wigner distribution can be seen to be a one-parameter level-spacing distribution that allows one to interpolate between chaotic and nearly integrable regimes.

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Over the past two decades random matrix theory has proven to be a useful tool in the study of quantum chaos. Universal signatures of quantum chaos were found for systems whose corresponding classical Hamiltonians exhibit chaotic behavior, and they are reviewed in [1, 2, 3]. The level spacing distributions derived from ensembles of random matrices [4, 5] have been shown to describe closely the statistics of fluctuations in the energy spectra of some chaotic systems [6]. This concept has also been extended to scattering [7, 8], dissipative systems [9], inter-series mixing in helium [10] and in general the analysis of systems with mixed regular and chaotic phase space regions [11].

In the generic case of mixed phase space, the Hamiltonian can be written as  $H = H_o + \epsilon H_1$ , with  $H_o$  an integrable Hamiltonian and  $H_1$  a non-integrable Hamiltonian. Regular and chaotic regions coexist in the mixed phase space and no universal laws have been shown to hold. Therefore level spacing distributions between neighboring eigenvalues have been suggested [3, 12, 13, 14] that are composed of the distributions describing the regular  $H_o$  and chaotic  $H_1$  regions separately. The regular regions have classically integrable Hamiltonians  $H_o$  whose eigenvalues are uniformly distributed and therefore the nearest-neighbor spacing statistics are described by Poisson distributions [12]. The eigenvalues of the non-integrable Hamiltonians  $H_1$  in the chaotic regions on the other hand are assumed to have an ergodic phase space region and are thus taken to be Gaussian distributed. The nearest-neighbor statistics for the chaotic regions then follow the Wigner-surmised level spacing statistics. These composite distributions then contain one or more adjustable parameters that allow for an interpolation between the Poisson and the Wigner level statistics.

The individual regions have Hamiltonians  $H_o, H_1$  whose eigenvalues are assumed to be independent. Thus an assumption of statistical independence of the random

matrix elements (the eigenvalues) is inherent to the theory also as a consequence of a semi-classical approximation. This assumption is admittedly invalid in even the simplest quantum case [13] if tunneling between the different regions is not neglected.

In this letter the level spacing distribution of a generic  $H$  is obtained from statistically dependent random matrix elements corresponding to correlated eigenvalues. The importance of using non-extensive statistic is that the nearest-neighbor level spacing distribution allows one to interpolate from the nearly integrable to the chaotic regime as the non-extensive parameter  $q$  is varied. The Hamiltonian matrix of  $H$  to be considered is an  $N \times N$  real symmetric matrix, and is invariant under orthogonal transformations.

The extensive Gaussian random matrix theory can be generalized by examining the non-extensive, or  $q$ -parameterized entropy [15, 16]. For systems with statistically dependent (say, Hamiltonian matrix elements) variables the joint probability decomposition is

$$P(H_i, H_j) = P(H_i | H_j)P(H_j), \quad (1)$$

which gives the pseudo-additive entropy

$$\begin{aligned} S_q(H_i, H_j) &= S_q(H_j) + S_q(H_i | H_j) \\ &+ (1 - q)S_q(H_j)S_q(H_i | H_j), \\ S_q &= -\ln_q P = -\frac{P^{1-q} - 1}{1 - q}. \end{aligned} \quad (2)$$

It is known [17] that the Tsallis entropy satisfies this condition, and the resulting probability will be of the power-law form. The  $q$ -logarithm is  $\ln_q X = -\frac{1 - X^{1-q}}{(q-1)}$ . In the limit as  $q \rightarrow 1$  the usual form of the natural logarithm and thus the extensive statistics and its exponential (Gaussian) distributions is recovered.

The entropy to be maximized given the constraints is then

$$\begin{aligned} \langle S \rangle_q &= -\frac{1 - \int P_N^q(H) dH}{(q-1)}, \\ \langle Tr(H^2) \rangle_q &= \int Tr(H^2) P_N^q(H) dH \end{aligned}$$

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$$= \sigma_q^2, \quad (3)$$

and which is subject to the extra normalization condition

$$\int P_N(H) dH = 1. \quad (4)$$

Then maximizing  $\langle S \rangle_q$  with the above constraints, yields the least biased probability density or non-extensive distribution function

$$P_N(H) = A_N (1 + \beta(q-1)Tr(H^2))^{\frac{-1}{q-1}}, \quad (5)$$

where  $A_N$  is a normalization constant and

$$Tr(H^2) = \sum_{1 \leq i \leq N} H_{ii}^2 + 2 \sum_{1 \leq i < j \leq N} H_{ij}^2. \quad (6)$$

This is then the Tsallis power-law form of the probability density for the  $N \times N$  random matrix elements. This nonextensive distribution then is use to obtain the correlated distribution function for the eigenenergies assuming that the given Hamiltonian is symmetric, real and can be diagonalized by means of an orthogonal transformation. The correlated distribution is given as

$$P_N(E_1, \dots, E_N) = A_N \prod_{j>i} |E_i - E_j| \left( 1 + \beta(q-1) \sum_i E_i^2 \right)^{\frac{-1}{q-1}}. \quad (7)$$

Consider a special case of non-extensive ensembles of  $2 \times 2$  matrices. The nearest neighbor spacing distribution  $p(s)$  is obtained from the correlated energy distribution function  $P(E_1, E_2)$  by

$$p(s) = \int_{-\infty}^{\infty} dE_1 \int_{-\infty}^{\infty} dE_2 P(E_1, E_2) \delta(s - |E_1 - E_2|) \\ p(s) = A \int_{-\infty}^{\infty} dE_1 \int_{-\infty}^{\infty} dE_2 |E_1 - E_2| \\ \times \left( 1 + \beta(q-1) \sum_i E_i^2 \right)^{\frac{-1}{q-1}} \delta(s - |E_1 - E_2|). \quad (8)$$

The constants  $A$  and  $\beta$  are fixed by the two normalization conditions  $\int p(s) ds = 1$ ,  $\int s p(s) ds = 1$ .

Integrating Eq.7. with the above conditions yields the non-extensive nearest neighbor spacing distribution.

$$p(s) = \frac{(5-3q)\beta}{2} s (1 + \beta(q-1)s^2/2)^{\frac{-1}{q-1} + \frac{1}{2}}, \quad (9)$$

where

$$\beta(q) = \frac{\pi}{2} \frac{1}{q-1} \left( \frac{\Gamma(\frac{1}{q-1} - 2)}{\Gamma(\frac{1}{q-1} - \frac{3}{2})} \right)^2, \quad (10)$$

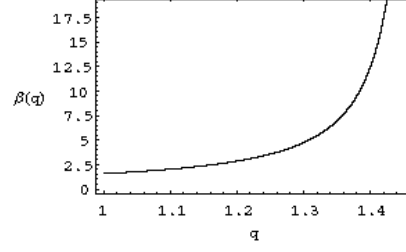


FIG. 1:  $\beta$  Vs.  $q$

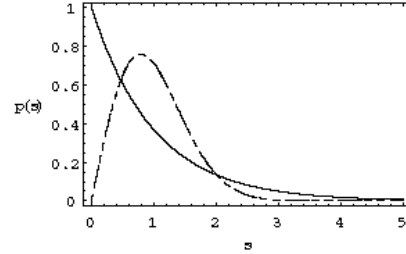


FIG. 2:  $P(s)$  Vs.  $s$ ,  $q = 1.01$ . The non-extensive and extensive Wigner distributions are nearly super-imposed. The Poisson distribution is plotted using a solid line.

and  $1 < q < \frac{3}{2}$ .

The behavior of the non-extensive nearest neighbor spacing distribution obtained by plotting its values for a range of the level spacing  $s$ , given the ‘inverse variance’  $\beta$  and the non-extensivity parameter  $q$ . In Fig.1. the  $q$  dependence of  $\beta$  is plotted for values of  $q$  between  $1 < q < 1.5$ . In Fig.2. the Poisson (solid), extensive (long-short dashed) and non-extensive Wigner (short dashed) distributions are plotted for a low non-extensivity parameter  $q$  value of  $q = 1.01$ . The non-extensive distribution is nearly superimposed on the extensive Wigner distribution as is expected for  $q- > 1$ . In Fig.3. the Poisson (solid), extensive (long-short dashed) and non-extensive Wigner (short dashed) are plotted for a high value of the non-extensivity parameter  $q = 1.38$ . Here the distribution is greatly shifted and approaches the Poisson level statistics distribution.

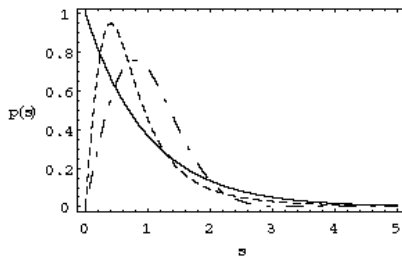


FIG. 3:  $P(s)$  Vs.  $s$ ,  $q = 1.38$ . The extensive distribution has been plotted with long-short dashing. The non-extensive Wigner distribution is plotted with the short dashes. The Poisson distribution is plotted using a solid line.

In this letter Gaussian distributed  $N \times N$  random Hamiltonian matrix elements is generalized to the case of the

non-extensive statistics and the resultant power-law distributions. A derivation of the subsequent level spacing statistical distribution, the non-extensive Wigner distribution, is given. This derivation is obtained from maximizing the non-extensive entropy of Tsallis of the  $N \times N$  symmetrical random matrix elements. The resultant of nonextensive level-spacing distribution is  $q$ -parameterized and the nearest neighbor spacing distribution varies from near integrable to chaotic regimes as the non-extensive parameter  $q$  is varied. The major importance is the possibility of using non-extensive statistics to shed light on the quantum signatures of classically mixed systems. In future work it will be interesting to apply these results to Hamiltonians of mixed systems between regular and chaotic regimes where deviations from the Wigner statistics become pronounced.

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